Lec 4: 01/30/2012

Special Relativity (Cont'd):

An important example of a four vector, as discussed, is the four momentum:

$$p^{\nu} = (E, c\vec{p})$$

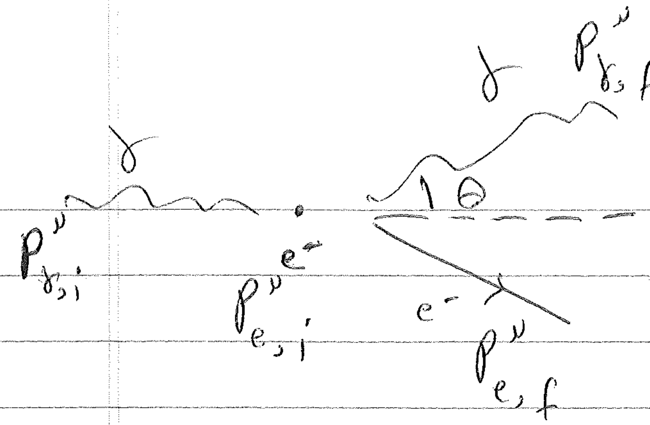
$$p^2 = p^{\nu} p_{\nu} = E^2 - c^2 |\vec{p}|^2 = (mc^2)^2$$

For a massless particle like photon we have:

$$E^2 - c^2 |\vec{p}|^2 = 0 \Rightarrow E = cp \qquad p = |\vec{p}|$$

The conservation of energy and momentum in classical mechanics can be combined into the conservation of four momentum in special relativity. The conservation law $P_i^{\nu} = P_f^{\nu}$ is held in all inertial frames.

Example: As a useful example, lets consider Compton scattering, which is scattering of a photon off an electron.



Assuming that the electron is initially at rest will make the calculation easier. The conservation of four-momentum during the scattering requires that:

$$p^\nu_{\gamma,i} + p^\nu_{e,i} = p^\nu_{\gamma,f} + p^\nu_{e,f} \Rightarrow p^\nu_{e,f} = p^\nu_{\gamma,i} + p^\nu_{e,i} - p^\nu_{\gamma,f} \Rightarrow$$

$$p^\nu_{e,f} = (p^\nu_{\gamma,i} - p^\nu_{\gamma,f}) + p^\nu_{e,i}$$

Notice that:

$$p^\nu_{e,i} = (m_e c^2, 0, 0, 0)$$

$$E_{\gamma,i} = c\,|\vec{p}_{\gamma,i}| \quad , \quad E_{\gamma,f} = c\,|\vec{p}_{\gamma,f}|$$

The conservation law implies that:

$$p^2_{e,f} = ((p_{\gamma,i} - p_{\gamma,f}) + p_{e,i})^2$$

$$p^2_{e,f} = (m_e c^2)^2$$

$$((p_{\gamma,i} - p_{\gamma,f}) + p_{e,i})^2 = p^2_{e,i} + (p_{\gamma,i} - p_{\gamma,f})^2 + 2\eta_{\nu\mu}(p_{\gamma,i} - p_{\gamma,f})^\nu p^\mu_{e,i}$$

$$= (m_e c^2)^2 + \cancel{P_{\gamma,i}^{2}}^{\,0} + \cancel{P_{\gamma,f}^{2}}^{\,0} - 2\eta_{\mu\nu} P_{\gamma,i}^{\mu} P_{\gamma,f}^{\nu} + 2(E_{\gamma,i} - E_{\gamma,f})^{m_e c^2}$$

$$= (m_e c^2)^2 - 2 E_{\gamma,i} E_{\gamma,f} + 2 \vec{P}_{\gamma,i} \cdot \vec{P}_{\gamma,f} + 2 m_e c^2 (E_{\gamma,i} - E_{\gamma,f})$$

$$= (m_e c^2)^2 - 2 E_{\gamma,i} E_{\gamma,f} + 2 E_{\gamma,i} E_{\gamma,f} \cos\theta + 2 m_e c^2 (E_{\gamma,i} - E_{\gamma,f})$$

Thus:

$$(m_e c^2)^2 = (m_e c^2)^2 - 2 E_{\gamma,i} E_{\gamma,f} (1 - \cos\theta) + 2 m_e c^2 (E_{\gamma,i} - E_{\gamma,f})$$

If the incident and scattered photons have frequencies ω, ω' respectively ($E_{\gamma,i} = \hbar\omega$, $E_{\gamma,f} = \hbar\omega'$), we will have:

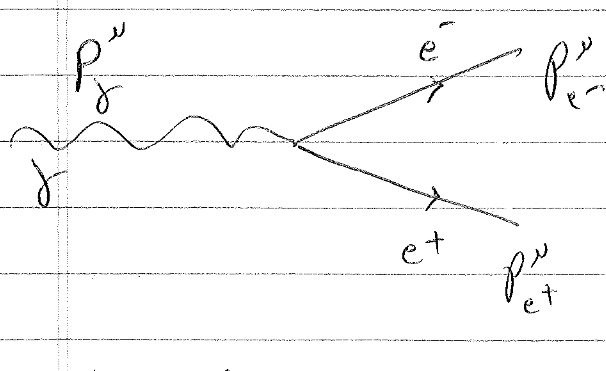
$$\cancel{\hbar}\,\omega\omega'(1-\cos\theta) = \cancel{2}\, m_e c^2 \hbar (\omega - \omega') \Rightarrow \omega'[m_e c^2 + \omega(1-\cos\theta)] = m_e c^2 \omega \Rightarrow \omega' = \frac{m_e c^2 \omega \hbar}{m_e c^2 + \omega(1-\cos\theta)} \Rightarrow \omega' = \frac{\omega}{1 + \frac{\hbar\omega}{m_e c^2}(1-\cos\theta)}$$

This result implies that $\omega' \leq \omega$. If $\theta = 0$ (forward scattering), we will have $\omega' = \omega$. While, for any angle $\theta \neq 0$, we find $\omega' < \omega$.

In the non-relativistic limit when $E_{\gamma,i} \ll m_e c^2$, we have $\omega' \simeq \omega$. This is the Thomson scattering limit.

This example deals with kinematic aspects of the Compton scattering. We will discuss its dynamic aspects and the physics behind it in detail later on. This is one of the most important interactions between high energy photons and free electrons.

Example: As another example, lets explore the possibility that a single photon can decay to a positron-electron pair.



Conservation of energy and momentum requires that:

$$p_\gamma^\nu = p_{e^-}^\nu + p_{e^+}^\nu \Rightarrow p_\gamma^2 = (p_{e^-} + p_{e^+})^2 \Rightarrow \cancel{p_\gamma^2}^{\,0} = p_{e^-}^2 + p_{e^+}^2$$

$$+ 2\eta_{\nu\nu'} \, p_{e^-}^\nu p_{e^+}^{\nu'} \Rightarrow m_e^2 + m_e^2 + 2\eta_{\nu\nu'} \, p_{e^-}^\nu p_{e^+}^{\nu'} = 0 \Rightarrow m_e^2 + E_{e^-} E_{e^+} -$$

$\vec{p}_{e^-} \cdot \vec{p}_{e^+} = 0$

Notice that $E_{e^+} \geq |\vec{p}_{e^+}|$ and $|E_{e^-}| \geq |\vec{p}_{e^-}|$. This implies that:

$$E_{e^-} E_{e^+} - \vec{p}_{e^-} \cdot \vec{p}_{e^+} \geq 0$$

Thus:

$$m_e^2 + E_{e^-} E_{e^+} - \vec{p}_{e^-} \cdot \vec{p}_{e^+} \geq m_e^2$$

Therefore the equality $m_e^2 + E_{e^-} E_{e^+} - \vec{p}_{e^-} \cdot \vec{p}_{e^+} = 0$ that is required by the energy and momentum conservation cannot hold. In consequence the process $\gamma \to e^- e^+$ does not happen.

The main reason here is the masslessness of the photon. This is strictly the case in the vacuum. In dense astrophysical media (such as supernova) the photon acquires a mass. In these cases, the decay to a positron-electron pair can proceed.